Extended and localized states of generalized kicked Harper models

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Basic properties of quantum states for generalized kicked Harper models are studied using the phase-space translational symmetry of the problem. Explicit expressions of the quasienergy (QE) states are derived for general rational values q/p of a dimensionless $\hbar$. The quasienergies form p bands and the QE states are q-fold degenerate. With each band one can associate a pair of integers σ and μ determined from the periodicity conditions of the QE states in the band. For $q = 1$, σ is exactly the Chern index introduced by Leboeuf *et al.* [Phys. Rev. Lett. **65**, 3076 (1990)] for a characterization of the classical-quantum correspondence. It is shown, however, that σ is *always different* from zero for $q > 1$. The Chern-index characterization is then generalized by introducing localized quantum states associated in a natural way with $\sigma = 0$. These states are formed from q QE bands with a total $\sigma = 0$ and they define q equivalent new bands, each with $\sigma = 0$. While these states are nonstationary, they become stationary in the semiclassical limit $p \to \infty$.

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I. INTRODUCTION

This paper is devoted to the study of basic properties of the general quantum Hamiltonian

$$H = W(v) + V(u) \sum_{s=-\infty}^{\infty} \delta(t/T - s) \ , \tag{1}$$

where u and v are dimensionless conjugate phase-space variables, $[u,\ v] = 2\pi i\rho$ (ρ is a constant), V and W are general periodic functions with period 2π, and T is the time period. The nonintegrable system (1) is exactly related [1,2] to the problem of periodically kicked charges in a uniform magnetic field under resonance conditions [3,4]. The case $W(v) = L\cos(v)$ and $V(u) = K\cos(u)$ (L and K are parameters), corresponding to the so-called kicked Harper (KH) model, has been extensively studied in the recent years [5–11]. This system exhibits a rich variety of quantum-dynamical properties, due to a very nontrivial quasienergy (QE) spectrum (i.e., the spectrum of the one-period evolution operator).

An interesting study of the KH model was performed in Ref. [5], in the framework of a toroidal phase space. In this framework, the QE spectrum always consists of a finite set of levels for each given boundary condition on the torus. By considering all the boundary conditions, a level "broadens" into a QE band. With each band one can associate a topological Chern invariant C (an integer), analogous to a quantized Hall conductance for a magnetic subband in the problem of Bloch electrons in a magnetic field [12]. Several arguments, supported by numerical evidence, indicate that in the semiclassical limit QE states with $C \neq 0$ are spread over the classical chaotic region (i.e., they are extended states). On the other hand, QE states with $C = 0$ are localized in this limit on regular orbits [Kol'mogorov-Arnol'd-Moser (KAM) tori or periodic orbits] or may be identified with scars [13], localized on unstable periodic orbits within the chaotic region.

In this paper, basic properties of quantum states for the "generalized" KH model (1) are systematically studied using the phase-space translational symmetry of the model. Some aspects of this symmetry have been considered recently [14, 15, 2] for the related problem of the kicked harmonic oscillator. Here the symmetry approach will be used to derive explicit expressions for the QE states and to study their properties. We shall consider general rational values of ρ, $\rho = q/p$ (q and p are relatively prime integers). The study of the KH model in Ref. [5] corresponds to the special case $q = 1$. Our results for $q > 1$ indicate that the Chern-index characterization of the classical-quantum correspondence [5] should not be restricted to QE states, but should be generalized to include nonstationary states.

We show that, in general, the QE spectrum consists of p bands and the QE states are q-fold degenerate. Each band can be characterized by an integer σ, determined from the periodicity conditions of the QE states in the band. This integer corresponds to the Chern topological invariant C in Ref. [5]. We find, however, a *second* Chern invariant μ for a band. The integers σ and μ are related by a Diophantine equation, from which it follows that σ can *never vanish* for $q > 1$. This fact is closely connected with the q-fold degeneracy of the QE states. Situations where all the Chern indices are nonzero have been pointed out by Faure and Leboeuf [5] in the case of integrable systems and have also been attributed to degeneracies due to "special" symmetries. However, the symmetry of the KH model for $q > 1$ is certainly not special. It is, in fact, the most general symmetry allowing for a toroidal-phase-space description. It is therefore most important to maintain, in this generic case, the clear distinction between localized and extended states on the basis of a zero or a nonzero value of the Chern index.

We show that localized states with $\sigma = 0$ for $q > 1$ can be consistently defined as linear combinations of QE states in a group of q bands with a total value of σ equal to zero (in general, the total value of σ for n bands may vanish only if n is a multiple of q). Each such group of QE bands provides q independent localized states defining q "new" bands, which are completely equivalent to the q QE bands. Each new band is associated with a zero value of σ. This is a natural definition of localized states, having the desirable property of being relatively stable under variations of ρ. While the localized states are necessarily nonstationary (even in a toroidal phase space), they become stationary in the semiclassical limit $p \to \infty$. We present here only the general theory. The detailed construction of quantum states properly localized on classical orbits for $q > 1$, as well as a study of their quantum dynamics, will be the subject of a separate publication [16].

Some of our results have analogs in the theory of Bloch electrons in uniform magnetic fields with $1/\rho$ flux quanta through a unit cell [17–20] (we shall refer to this system as to system M). In fact, system M is also invariant under phase-space translations, i.e., the "magnetic translations" [17]. However, the results obtained for (1) usually contain more information than the corresponding results for system M. This is because system (1) features only *one* degree of freedom $[(u,\ v)]$, in contrast with system M featuring two degrees of freedom.

The paper is organized as follows. In Sec. II we consider the phase-space translational symmetry of (1) for general rational values q/p of ρ and derive from it explicit expressions for the QE states. Basic properties of these states are studied in Sec. III, after reducing the QE problem to the eigenvalue problem of a $p \times p$ matrix. The QE states are then characterized by two Chern integers σ and μ, satisfying a Diophantine equation and sum rules. In Sec. IV we generalize the Chern-index characterization of Ref. [5] by introducing phase-space lattices of localized states spanning q bands. The conditions for the completeness and orthogonality of these states relative to the space of the q bands are derived. The integrable limit of the Harper model [21, 12] is discussed. Conclusions are presented in Sec. V.

II. PHASE-SPACE TRANSLATIONAL SYMMETRY AND QUASIENERGY STATES

The one-period evolution operator for (1), from $t = -0$ to $t = T - 0$, is given by

$$U = \exp[-iW(v)T/\hbar]\ \exp[-iV(u)T/\hbar]\ . \qquad (2)$$

Because of $[u,\ v] = 2\pi i\rho$ and the 2π periodicity of V and W, the operator (2) commutes with the phase-space translations

$$D_0 = e^{i\alpha v}\ , \qquad D_1 = e^{i\alpha u}\ , \qquad (3)$$

with the *minimal* value of $\alpha = 1/\rho$. The operators (3) generate then the invariance group G of U, which is a subgroup of the Heisenberg-Weil group W_1 [22] of phase-space translations in the $(u,\ v)$ plane. In general, the operators (3) ($\alpha = 1/\rho$) do not commute, but G may have Abelian (commutative) subgroups G_a with basic elements D_0^r and D_1^s, for some integer powers r and s. This is the case only if ρ is rational:

$$\frac{rs}{\rho} = p \quad \Longrightarrow \quad \rho = \frac{q}{p}\ , \qquad (4)$$

where $q = rs$ and q and p are coprime if we choose the smallest value of rs satisfying (4). The operators D_0^r and D_1^s generate then a maximal Abelian subgroup G_a of G. From now on, we shall make the "standard" choice $r = q$, $s = 1$, giving the basic commuting operators

$$D_1 = e^{iu/\rho}\ , \qquad D_2 \equiv D_0^q = e^{ipv}\ . \qquad (5)$$

The QE states will be chosen as simultaneous eigenstates of the operators (2) and (5). Since G can be expressed as the coset sum $\sum_{r=0}^{q-1} D_0^r G_a$, each QE state must be *q-fold degenerate* (see also below).

In order to derive explicit expressions for the QE states, we introduce first a quantum-mechanical representation based on the complete set of operators $\bar{D}_1 \equiv e^{i\beta u}$ and D_2, where β is the smallest number for which $\bar{D}_1$ and D_2 commute [23]. We easily find that $\beta = 1/(\rho p) = 1/q$, so that the complete set is

$$\bar{D}_1 = e^{iu/q}\ , \qquad D_2 = e^{ipv}\ . \qquad (6)$$

The operators (6) generate a maximal Abelian subgroup W_a of the Heisenberg-Weil group. The simultaneous eigenfunctions of (6), defining the representation, are given by the kq distributions [23]

$$\psi_{\mathbf{w}}(v) = \sum_{l=-\infty}^{\infty} \exp(ilw_1/q)\delta(v - w_2 + 2\pi l/p)\ , \qquad (7)$$

where $\mathbf{w} = (w_1,\ w_2)$ is the "quasimomentum," giving the eigenvalues $\exp(iw_1/q)$ and $\exp(ipw_2)$ of $\bar{D}_1$ and D_2, respectively, and ranging in the "Brillouin" zone

$$0 \le w_1 < 2\pi q\ , \qquad 0 \le w_2 < 2\pi/p\ . \qquad (8)$$

Since $D_1 = \bar{D}_1^p$, the group G_a is a subgroup of index p of W_a and a general eigenstate of the operators (5) can then be expressed as a linear combination of p distributions (7) at $\mathbf{w} = (w_1 + 2\pi mq/p,\ w_2)$, $m = 0, \ldots,\ p-1$. In particular, this will be true for the QE states, which are simultaneous eigenstates of (2) and (5):

$$\Psi_{b,\mathbf{w}}(v) = \sum_{m=0}^{p-1} \phi_b(m;\ \mathbf{w})\psi_{w_1+2\pi m\rho,w_2}(v)\ , \qquad (9)$$

where b is a "band" index (see below), which should assume precisely p values, $b = 1, \ldots,\ p$, corresponding to p independent vectors of coefficients $\phi_b(m;\ \mathbf{w})$, which are needed to obtain a complete set of functions (9). The states (9) are eigenstates of (5) with eigenvalues $\exp(iw_1/\rho)$ and $\exp(ipw_2)$, where $\mathbf{w}$ now ranges in the reduced Brillouin zone

$$0 \le w_1 < 2\pi\rho\ , \qquad 0 \le w_2 < 2\pi/p\ , \qquad (10)$$

which is $1/p$ of the zone (8).

III. PROPERTIES OF QE STATES AND CHERN INDICES

We now require (9) to be eigenstates of U in (2) with eigenvalues $\exp(-iE_bT/\hbar)$, where E_b are the quasienergies. Expanding in the Fourier series

$$\exp[-iV(u)T/\hbar] = \sum_{r=-\infty}^{\infty} J_{1,r}e^{iru} ,$$

$$\exp[-iW(v)T/\hbar] = \sum_{r=-\infty}^{\infty} J_{2,r}e^{irv} , \tag{11}$$

where $J_{1,r}$ and $J_{2,r}$ are "generalized" Bessel functions, and applying (11) to (9), we find, after some algebra,

$$U\Psi_{b,\mathbf{w}}(v) = \sum_{l,\ l'=0}^{p-1} F_{1,l}(w_1)F_{2,l'}(w_2)e^{i(lw_1+l'w_2)} \times \sum_{m=0}^{p-1} \phi_b(m;\ \mathbf{w})e^{2\pi ilm\rho}\psi_{w_1+2\pi(m-l')\rho,w_2}(v) , \tag{12}$$

where, for $j = 1, 2$,

$$F_{j,l}(w_j) = \sum_{r=-\infty}^{\infty} J_{j,rp+l}e^{irpw_j} . \tag{13}$$

Defining

$$\bar{\phi}_b(m;\ \mathbf{w}) \ \equiv\ e^{imw_2}\phi_b(m;\ \mathbf{w}) \tag{14}$$

and using the independence of the kq functions (7) [23], we obtain from (12) the eigenvalue problem for the vector $V_b(\mathbf{w}) \equiv \{\bar{\phi}_b(m;\ \mathbf{w})\}_{m=0,\dots,p-1}$:

$$\hat{M}(\mathbf{w})V_b(\mathbf{w}) = \exp[-iE_b(\mathbf{w})T/\hbar]V_b(\mathbf{w}) , \tag{15}$$

where the $p \times p$ matrix $\hat{M}(\mathbf{w})$ has elements

$$\hat{M}_{m,m'}(\mathbf{w}) = \exp[-iV(w_1+2\pi m'\rho)T/\hbar]F_{2,m'-m}(w_2) , \tag{16}$$

$m,\ m' = 0,\dots,\ p-1$. Thus, at fixed $\mathbf{w}$, the QE spectrum consists of p levels E_b, which span p QE bands $E_b(\mathbf{w})$ as $\mathbf{w}$ varies in the zone (10). Now, the eigenvalue problem (15) with (16) and (13) is clearly periodic in the two-dimensional torus

$$0 \le w_1 < 2\pi , \qquad 0 \le w_2 < 2\pi/p . \tag{17}$$

On the other hand, it is easy to show, using (14) and (16), that the eigenvalue spectrum is invariant under $w_1 \to w_1 + 2\pi\rho$, with the set of eigenvectors $V_b(\mathbf{w})$ satisfying

$$V_b(w_1 + 2\pi\rho,\ w_2) = \exp[i\gamma_b(\mathbf{w})]\{\exp[iw_2(m - |m+1|_p)] \times\bar{\phi}_{b'}(|m+1|_p;\ \mathbf{w})\}_{m=0,\dots,p-1} , \tag{18}$$

where $\exp[i\gamma_b(\mathbf{w})]$ is some phase factor, $b' = b'(b)$, and $|m|_p \equiv m$ mod$p$. Since the eigenvalue spectrum is periodic in w_1 with the periods $2\pi\rho$ and 2π [see (17)], the actual period of the spectrum must be $2\pi/p$.

We now make the assumption of *noncrossing* QE bands, i.e., that $\exp\{i[E_{b'}(\mathbf{w}) - E_b(\mathbf{w})]T/\hbar\} \neq 1$ for $b \neq b'$. It then follows that $\exp[-iE_b(\mathbf{w})T/\hbar]$ is periodic in both w_1 and w_2 with the period $2\pi/p$, for each band b. Similarly, the relation (18) is satisfied with $b' = b$ and $V_b(\mathbf{w})$ is periodic in w_2 with period $2\pi/p$ (up to some phase factor). This implies that the QE states (9) are periodic in the zone (10), but, in general, only up to a $\mathbf{w}$-dependent phase factor. As in the case of magnetic Bloch states [18, 19], the phase of (9) can always be chosen so that the periodicity conditions satisfied by the QE states are, in general,

$$\Psi_{b,w_1+2\pi\rho,w_2}(v) = \Psi_{b,\mathbf{w}}(v) , \tag{19}$$

$$\Psi_{b,w_1,w_2+2\pi/p}(v) = \exp(i\sigma_b w_1/\rho)\Psi_{b,\mathbf{w}}(v) . \tag{20}$$

Here σ_b is an integer corresponding precisely to the Chern index C in Ref. [5] (see Sec. IV). The choice of phase of the QE states is then such that $\gamma_b(\mathbf{w}) = 0$ in the relation (18) (with $b' = b$). From the fact that the eigenvalues $\exp[-iE_b(\mathbf{w})T/\hbar]$ are periodic in a zone q times smaller than (10) (see above), it follows that the QE states in each band are *q-fold degenerate*. The degenerate states may be obtained by applying to $\Psi_{b,\mathbf{w}}(v)$ the q operators D_0^r, $r = 0,\dots,\ q-1$, where $D_0 = e^{iv/\rho}$ commutes with U (see Sec. II). Using (9) with (7) and the fact that $V_b(w_1+2\pi,\ w_2) = V_b(\mathbf{w})$ [see the relation (24) below, for which we show that $\chi_{b,1}(\mathbf{w}) = 0$], we get

$$D_0^r\Psi_{b,\mathbf{w}}(v) = \exp(irw_2/\rho)\Psi_{b,w_1-2\pi r,w_2}(v). \tag{21}$$

Besides the Chern index σ_b in (20), there is a *second* Chern index for the problem. In fact, the periodic matrix (16) can be characterized by p Chern homotopic invariants [24], i.e., integers μ_b associated with the p bands:

$$\mu_b = \frac{i}{2\pi}\oint V_b^\dagger(\mathbf{w})\frac{dV_b(\mathbf{w})}{d\mathbf{w}}\cdot d\mathbf{w} , \tag{22}$$

where the contour integral is taken around the boundary of the torus (17). A useful formula for the computation of μ_b can be derived using Stoke's theorem in (22) and expressions for $dV_b(\mathbf{w})/d\mathbf{w}$ obtained by differentiating both sides of Eq. (15) with respect to $\mathbf{w}$:

$$\mu_b = \frac{i}{2\pi}\int\int d\mathbf{w}\sum_{b'\neq b}\frac{V_b^\dagger\frac{\partial\hat{M}^\dagger}{\partial w_1}V_{b'}V_{b'}^\dagger\frac{\partial\hat{M}}{\partial w_2}V_b - V_b^\dagger\frac{\partial\hat{M}^\dagger}{\partial w_2}V_{b'}V_{b'}^\dagger\frac{\partial\hat{M}}{\partial w_1}V_b}{|e^{iE_{b'}} - e^{iE_b}|^2} , \tag{23}$$

where the double integral is taken over the torus (17).

We now derive a relation between the Chern indices σ_b and μ_b. Consider the general periodicity conditions of the eigenvectors $V_b(\mathbf{w})$ in the torus (17):

$$V_b(w_1 + 2\pi,\ w_2) = \exp[i\chi_{b,1}(\mathbf{w})]V_b(\mathbf{w}) , \tag{24}$$

$$V_b(w_1,\ w_2 + 2\pi/p) = \exp[i\chi_{b,2}(\mathbf{w})]V_b(\mathbf{w}) . \tag{25}$$

The phase $\chi_{b,1}(\mathbf{w})$ in (24) can be consistently chosen to be zero. This can be seen by iterating the relation (24) q times and comparing the result with the pth iteration of the relation (18) [$\gamma_b(\mathbf{w}) = 0$ and $b' = b$]. The phase $\chi_{b,2}(\mathbf{w})$ in (25) can be determined as follows. From Eq. (20) with (9), (7), and (14), we get

$$V_b(w_1,\ w_2 + 2\pi/p) = \exp(-iw_1/q + i\sigma_b w_1/\rho) V_b(\mathbf{w}) \ . \tag{26}$$

By comparing (26) with (25), we find that $\chi_{b,2}(\mathbf{w}) = \xi_b w_1$, where $\xi_b = \sigma_b/\rho - 1/q$. Now, this form of $\chi_{b,2}(\mathbf{w})$ is consistent with the relation (24) [$\chi_{b,1}(\mathbf{w}) = 0$] only if ξ_b is an integer. This integer is precisely the Chern index μ_b, as one can easily verify using (24) and (25) (with the phases above) in (22). The expression above for ξ_b can then be written as

$$p\sigma_b - q\mu_b = 1 \ . \tag{27}$$

The Diophantine equation (27) allows one to determine uniquely σ_b once μ_b is known [e.g., from formula (23)]. It follows from (27) that when $q > 1$, σ_b cannot assume, in principle, all values, but only values differing from each other by a multiple of q. Moreover, the value $\sigma_b = 0$ is clearly forbidden when $q > 1$. These properties are closely connected with the q-fold degeneracy of the QE states (see the Appendix).

Since the integers μ_b are homotopic invariants characterizing a finite $p \times p$ periodic matrix, one must have $\sum_{b=1}^{p} \mu_b = 0$ [24]. It then follows from Eq. (27) that

$$\sum_{b=1}^{p} \sigma_b = 1 \ . \tag{28}$$

In particular, for $p = 1$ (one-band QE spectrum), one must have $\mu_b = 0$ and $\sigma_b = 1$. In fact, the QE states (9) coincide in this case with the kq functions (7), which satisfy the periodicity conditions (19) and (20) with $\sigma_b = 1$. The one-band QE spectrum is explicitly given by

$$E_b(\mathbf{w}) = V(w_1) + W(w_2) \ .$$

IV. LOCALIZED STATES

A topological characterization of the classical-quantum correspondence was introduced in Ref. [5] in the framework of a toroidal phase space. In this phase space, an arbitrary quantum state has to be an eigenstate of two commuting phase-space translations defining the torus. The torus area must then be an integer multiple N of $2\pi\hbar$. The eigenvalues of the commuting phase-space translations determine the boundary conditions of the quantum state. A quantum-dynamical evolution operator assumes, for each boundary condition, precisely N eigenvalues that span N bands as the boundary conditions are varied. With each band one can associate a Chern index C determined from the periodicity conditions of the eigenstates in the band. Several arguments have been given that in the semiclassical limit eigenstates with $C \neq 0$ are spread over the classical chaotic region (extended states). On the other hand, eigenstates with $C = 0$ are localized in this limit on regular orbits (KAM tori or periodic orbits) or may be identified with scars [13], localized on unstable periodic orbits within the chaotic region.

This theory was illustrated in the case of the KH model, where the unit cell of periodicity was naturally identified with the torus (the phase space). One can easily verify that with this identification the basic commuting phase-space translations are given by (5) with $q = 1$ ($\rho = 1/p$). Moreover, with $\hbar = 2\pi\rho$, one finds that $N = p$. The study of the KH model in Ref. [5] is then essentially a special case of our approach. Using the formula determining the Chern index C from the periodicity conditions of the eigenstates in Ref. [5], one can easily ascertain that C is just the integer σ_b in (20) (see remark [25]). However, the existence of the second Chern index μ_b was not noticed in Ref. [5].

In the general case of $q > 1$, the toroidal phase space contains precisely q unit cells (and $N = p$ as before). In this case, the theory of Ref. [5] is not applicable since, as we have shown in Sec. III, $\sigma = C$ *never* vanishes. This fact is closely connected with the q-fold degeneracy of the QE states (see the Appendix). We now show, however, that quasistationary localized states with $\sigma = 0$ can be defined in a natural and consistent way for $q > 1$. This definition has the desirable property of being relatively stable under perturbations of ρ.

Consider a general square-integrable state, expressed as a linear combination of QE states belonging to an arbitrary set of n bands, $n \leq p$:

$$A_r(v) = \sum_{b=1}^{n} \int d\mathbf{w} B_b^{(r)}(\mathbf{w}) \Psi_{b,\mathbf{w}}(v) \ , \tag{29}$$

where $r = 1, \ldots, q$ is an index associating the state to one of the q unit cells (see below), the integral is taken over the Brillouin zone (10), and $B_b^{(r)}(\mathbf{w})$ is the expansion coefficient. The state (29) can be translated in phase space so as to be localized around an arbitrary unit cell:

$$A_r(v;\ l_1, l_2) = D_1^{l_1} D_2^{l_2} A_r(v) \ , \tag{30}$$

where l_1 and l_2 are integers and D_1 and D_2 are basic commuting translations, such as those in (5). In the toroidal phase space defined by D_1 and D_2, the quantum states corresponding to (30) should satisfy boundary conditions determined by $\mathbf{w}$ and should approach locally the states (30) as the size of the torus tends to infinity (e.g., $p, q \to \infty$). These properties are exhibited in a natural way by the symmetry-adapted sums

$$\bar{A}_r(v;\ \mathbf{w}) = \frac{p^2}{4\pi^2 q} \sum_{l_1, l_2 = -\infty}^{\infty} \exp(-iw_1 l_1/\rho - iw_2 l_2/p) \times A_r(v;\ l_1, l_2) = \sum_{b=1}^{n} B_b^{(r)}(\mathbf{w}) \Psi_{b,\mathbf{w}}(v) \ , \tag{31}$$

where the last expression follows from (30) and (29). In a sense, the index $\mathbf{w}$ in the toroidal phase space plays the role of the lattice index (l_1, l_2), labeling the states (30) in ordinary phase space. The relation (31) and its inverse,

$$A_r(v;\ l_1,l_2) = \int d\mathbf{w} \exp(iw_1 l_1/\rho + iw_2 l_2/p)\bar{A}_r(v;\ \mathbf{w}) \ ,$$

provide the connection between the two sets of states (30) and (31) in the ordinary and the toroidal phase space, respectively. A key observation is now that the states (31) satisfy the periodicity conditions (19) and (20) with $\sigma = 0$. One can then use the expression (31) as a starting point to define localized states. For this definition to be consistent with the one based on QE states with $C = 0$ in the case $q = 1$ [5], the n bands and the expansion coefficient $B_b^{(r)}(\mathbf{w})$ in (29) should be properly specified. The basic specification will be made here using general arguments. A more detailed specification of quantum states associated with classical orbits of (1) for $q > 1$ will be given elsewhere [16].

Our first observation is that rational values of ρ of the form $\rho = 1/p'$ and $\rho = q/p$ can be arbitrarily close to each other so that they should not lead to drastically different physical situations (stability under perturbations of ρ). It is reasonable to assume that each of the p' QE bands for $\rho = 1/p'$ should correspond, in some sense, to a group of approximately q adjacent bands in the case of $\rho = q/p$. Since the total value of σ is always 1 [see (28)], it is plausible to conjecture that the total value of σ for this group of bands should be approximately equal to the value of σ for the corresponding band in the case of $\rho = 1/p'$.

Let us check this conjecture in the integrable limit [$T \rightarrow 0$ in (1)] of the Harper model [21, 12], with $H = V(u) + W(v)$. For particular choices of V and W, one finds [12] that the total value of σ carried by the lowest (or highest) n bands, $n < p/2$, is uniquely determined by the equation $p\sigma + qs = n$, where the integer s must satisfy $|s| \leq p/2$. It follows from this that $\sigma = 0$ if n is a multiple of q. If we assume, for definiteness, that $p = 2lq + q'$, where l and q' are some positive integers ($q' < q$), the spectrum consists then of a "central" group of q' bands carrying a total $\sigma = 1$, surrounded by $2l$ groups of q bands, each group carrying a total $\sigma = 0$. The central group may be viewed as corresponding to the central band in the case of $\rho = 1/(2l + 1)$, while the surrounding groups correspond to the other $2l$ bands.

In general, it follows from Eq. (27) that a total value of $\sigma = 0$ may be carried only by a group of *precisely* q (or a multiple of q) bands. It is then natural to choose the q localized states (31) as linear combinations of a group of $n = q$ adjacent QE bands carrying a total $\sigma = 0$ (we assume, of course, that $q < p$). These states, characterized each by $\sigma = 0$ (see above), should then resemble the states associated with the corresponding band with $\sigma = 0$ in the case of $\rho = 1/p' \approx q/p$. In particular, for the Harper model discussed above, the states associated with any of the $2l$ groups of q bands should be localized on regular classical orbits, while the QE states in the central group of q' bands should be supported by the separatrix, as in the $\rho = 1/p'$ case [5].

The requirement to have $n = q$ bands in (29) with a total $\sigma = 0$ is actually most essential from a different point of view. The states (31) in the toroidal phase space are extended states in ordinary phase space. They may be viewed as defining q "new" bands, labeled by the index $r = 1,\ldots,\ q$, and corresponding, as a single entity, to a band with $\sigma = 0$ in the case of $\rho = 1/p' \approx q/p$. If these new bands are now to be used instead of the n original QE bands, one must necessarily have $n = q$. Moreover, since each new band in (31) is associated with $\sigma = 0$, it is intuitively obvious that the total value of σ, σ_q, carried by the original q QE bands must be zero [notice that Eq. (27) implies only that σ_q is a multiple of q]. In fact, we now show that the q new bands are completely equivalent to the original QE bands, i.e., they span the same space of functions if and only if $\sigma_q = 0$.

We first show that if $\sigma_q \neq 0$ the two sets of bands in (31) cannot be equivalent, since the $q \times q$ matrix $B_b^{(r)}(\mathbf{w})$, $b, r = 1,\ldots,\ q$, is not invertible for *all* $\mathbf{w}$. The function $B_b^{(r)}(\mathbf{w})$ clearly satisfies the periodicity conditions (19) and (20) with σ_b replaced by $-\sigma_b$. Moreover, this function must be, at least, continuous if the states (29) are sufficiently localized. The determinant $\Delta_q(\mathbf{w})$ of $B_b^{(r)}(\mathbf{w})$ is then, at least, a continuous function satisfying the conditions (19) and (20) with σ_b replaced by $-\sum_{b=1}^{q}\sigma_b = -\sigma_q$. It is known (see, e.g., the Appendix in Ref. [19]) that such a function must assume at least $|\sigma_q|$ zeros (counting multiplicities) in the Brillouin zone (10). This means that the transformation in (31) is not invertible for all $\mathbf{w}$ if $\sigma_q \neq 0$. On the other hand, when $\sigma_q = 0$, the matrix $B_b^{(r)}(\mathbf{w})$ can always be chosen to be nonsingular, in fact unitary. This is accomplished by choosing $B_b^{(r)}(\mathbf{w})$, $r = 1,\ldots,\ q$, as the q orthonormal eigenvectors of a $q \times q$ periodic Hermitian matrix whose q homotopic invariants are precisely the given σ_b's. Such a matrix can always be constructed explicitly [24]. This completes the proof of the claim above.

While the localized states (31) are nonstationary, their completeness and orthogonality properties are time independent. This is clear from the fact that after s time steps, the expansion coefficients in (31) are replaced by

$$B_b^{(r)}(\mathbf{w},\ t = sT) = \exp[-iE_b(\mathbf{w})sT/\hbar]B_b^{(r)}(\mathbf{w}) \ . \qquad (32)$$

Thus, for example, the determinant of the matrix (32) will never vanish if it does not vanish at $t = 0$. This expresses the simple fact that the zero Chern index characterizing the localized states (31) is a constant of the motion, despite the nonstationarity of these states. In any case, for q/p sufficiently small, the group of q QE bands in (32) will generally be very narrow and the localized states are then almost stationary.

By choosing the matrix $B_b^{(r)}(\mathbf{w})$ to be unitary in the case $\sigma_q = 0$ (see above), the localized states (31) become orthonormal, both in $\mathbf{w}$ and in the "new band" index r ("Wannier functions"). Then, different values of r should correspond, in general, to states $\bar{A}_r(v;\ \mathbf{w})$, $r = 1,\ldots,\ q$, localized around different unit cells within the torus. Thus the index r may label also the localization site. However, there is no reason to assume that the localization profiles around the q unit cells will be the same. Because of this fact, the Wannier functions may not be best localized on classical orbits in the semiclassical limit. A more natural choice is

$$\bar{A}_r(v;\ \mathbf{w}) = D_0^r \bar{A}(v;\ \mathbf{w})\ , \tag{33}$$

where $D_0 = e^{iv/\rho}$ is the basic phase-space translation within the torus (see Sec. II) and $\bar{A}(v;\ \mathbf{w})$ is some properly chosen state localized around the unit cell $r = 0$. Then $\bar{A}_r(v;\ \mathbf{w})$ in (33) is localized around the unit cell r and has the same localization profile as $\bar{A}(v;\ \mathbf{w})$. It is easy to show, using (21), that the expansion coefficients in (31), corresponding to the states (33), are given by

$$B_b^{(r)}(\mathbf{w}) = \exp(irw_2/\rho) B_b(w_1 + 2\pi r,\ w_2)\ , \tag{34}$$

where $B_b(\mathbf{w})$ is the expansion coefficient for $\bar{A}(v;\ \mathbf{w})$. In general, the matrix (34) has the desired property of being nonsingular for $\sigma_q = 0$, but it is not unitary. Thus the states (33) are not orthogonal, which is the price one must pay in order to have states naturally localized with the same localization profile on all unit cells.

It is instructive to compare the localized states (33) with the degenerate QE states (21), which are also generated by q applications of D_0. While $\sigma_b \neq 0$ for $q > 1$, a QE state $\Psi_{b,\mathbf{w}}(v)$ may be localized, in the Husimi (coherent-state) representation $\varphi_{b,\mathbf{w}}(z) = \langle z|\Psi_{b,\mathbf{w}}\rangle$, on classical orbits in one unit cell. This has been verified by Faure and Leboeuf [5] in the case of integrable systems for which all the Chern indices are nonzero. A degenerate QE state (21) will be then localized with the same localization profile in another unit cell. In the localization domain of $\Psi_{b,\mathbf{w}}(v)$, $\varphi_{b,\mathbf{w}}(z)$ does not vanish. However, since $\sigma_b \neq 0$, it may well vanish in regions corresponding to the localization domains of the degenerate states. In fact, this has been observed in the systems studied by Faure and Leboeuf [5]. Consider, on the other hand, the Husimi representation $\langle z|\bar{A}_r(v;\ \mathbf{w})\rangle$ of the localized states (33). Since $\sigma = 0$ for these states, it is possible to define their localization domain, as in Ref. [5], as the set of z values for which $\langle z|\bar{A}_r(v;\ \mathbf{w})\rangle$ never vanishes as $\mathbf{w}$ spans the Brillouin zone (10). It is then easy to show, using (33) and (34), that this localization domain repeats itself on all the q unit cells (i.e., it is translationally invariant), and it is the same for all the q localized states (33). At the same time, a state $\bar{A}_r(v;\ \mathbf{w})$ is, of course, dominantly localized in the rth unit cell. These properties, which are not possessed by the QE states for $q > 1$, are another indication of good localization features of the states (33). In a future work [16], we shall show how the basic function $B_b(\mathbf{w})$ in (34) should be chosen in order to give localized states (33) associated in a most natural way with classical orbits in the semiclassical limit.

V. CONCLUSIONS

The generalized KH models (1) form the simplest class of nonintegrable systems (1.5 degrees of freedom) exhibiting phase-space translational symmetry. They are also exactly related [1, 2] to the physically realizable system of periodically kicked charges in a uniform magnetic field under resonance conditions [3, 4]. We have used the phase-space translational symmetry of (1) to derive explicit expressions for the QE states and to study their basic properties. General rational values q/p of ρ (the dimensionless $\hbar$) have been assumed. These are the only values allowing for a toroidal-phase-space approach to the problem. The main properties established are that (a) the QE spectrum consists exactly of p bands, (b) the QE states are q-fold degenerate, (c) with each QE band b one can associate two Chern indices σ_b and μ_b related by a Diophantine equation, and (d) the sum of σ_b over the p bands is 1.

Properties (b) and (c) have analogs in the theory of Bloch electrons in magnetic fields with p/q flux quanta through a unit cell (system M) [18, 19]. The magnetic Bloch states are also q-fold degenerate and the Chern integer σ_b corresponds to the quantum Hall conductance carried by a magnetic band b [12]. Properties (a) and (d) do not hold exactly for system M, but only approximately in the limit of a strong magnetic field or weak periodic potential [19]. In this limit, a Landau level splits into p magnetic subbands and the relation (28) [property (d)] then means that the total Hall conductance carried by these subbands is equal to that of the Landau level. While the splitting of a Landau level into subbands has a sound group-theoretical basis [19, 26], it is not exact and holds only approximately in the limits above [20]. This is because system M features two degrees of freedom (the kinetic momentum and the guiding center), in contrast with system (1), featuring only the $(u,\ v)$ degree of freedom that generates the phase-space translational symmetry. As a consequence, some results that are always exact for system (1), e.g., formula (23) for the Chern index μ_b, hold only approximately for system M in the limits above. Moreover, the QE states can be expressed in a more closed form than the magnetic Bloch states [19].

In the generic case of $q > 1$, all the Chern indices σ_b are nonzero, so that according to the theory in Ref. [5] all the QE states should be considered as "extended." We have shown, however, that localized states with $\sigma = 0$ can be consistently defined as linear combinations of QE states in q bands: Each group of q QE bands with a total $\sigma = 0$ provides q independent localized states, which, as the boundary conditions $\mathbf{w}$ are varied, span q new bands completely equivalent to the QE bands. Each new band is associated with a zero value of σ. By arguments of stability under variations of ρ, the q QE bands are expected to be adjacent and to correspond to a QE band with $\sigma = 0$ for a neighboring value of ρ with $q = 1$. Then, while the localized states are nonstationary, they become stationary in the semiclassical limit $p \to \infty$. A general and natural form of the localized states is given by (33). These ideas will be extensively used in a future work [16] to construct explicitly quantum states properly localized on classical orbits and to study their quantum dynamics due to their nonstationarity for finite p.

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APPENDIX: DEGENERACY AND NONZERO CHERN INDICES

We show here, using also ideas in Ref. [5], how the q-fold degeneracy of the QE states is connected with the nonvanishing of all Chern indices σ_b for $q > 1$. The coherent-state representation of a QE state $\Psi_{b,\mathbf{w}}(v)$, $\varphi_{b,\mathbf{w}}(z) = \langle z|\Psi_{b,\mathbf{w}}\rangle$, assumes precisely $N = p$ zeros within the torus [i.e., the region $2\pi q \times 2\pi$ in the (u, v) phase plane] [5]. The degenerate states (21) have their zeros uniformly shifted by r unit cells (in the u direction), relative to the zeros of $\varphi_{b,\mathbf{w}}(z)$. Now, using Eq. (21), it is easy to show that degenerate states with minimal separation $2\pi/p$ in the Brillouin zone (see Sec. III) are given by $\Psi_{b,\mathbf{w}}(v)$ and $D_0^r\Psi_{b,\mathbf{w}}(v)$, where r is the smallest non-negative integer satisfying $r = (sq+1)/p$ (s integer). Clearly, $r \neq 0$ for $q > 1$. Thus every shift $w_1 \to w_1 - 2\pi/p$ causes the zeros of $\varphi_{b,\mathbf{w}}(z)$ to be shifted uniformly by r unit cells. After q such shifts of w_1, the Brillouin zone is completely spanned and the zeros wind precisely r times around the torus. Since $r \neq 0$, this winding of the zeros is an expression of the fact that $C = \sigma_b$ cannot vanish, as explained in Ref. [5]. In fact, by comparing the definition above of r with Eq. (27), we see that $r = \sigma_b \bmod q$.

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